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By

J. P. deVries
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SUMMARY

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This report is submitted in partial fulfillment of the contract in "Space Flight and Guidance Theory," No. NAS8-11040. It presents a discussion of Lagerstrom and Kevorkian's two-variable expansion method for the computation of lunar trajectories. Section 2 discusses the general background of the method in terms of singular perturbation theory. Section 3 discusses the major steps in the development of a uniformly valid solution for earth-moon trajectories and Section 4 presents a slightly different approach to the same problem.

Author

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1. INTRODUCTION

In refs. (3) and (4) a new method was suggested by Lagerstrom and Kevorkian for the computation of lunar trajectories. The method was similar to one which had been used successfully in a number of singular perturbation problems of boundary layer theory (refs. 1, 2). The result of approaching the lunar trajectory problem as a singular perturbation problem was a uniformly valid solution (i.e. valid everywhere in the earth-moon space) to first order in the parameter for a certain class of trajectories. The class of trajectories is that which starts in a neighborhood of order μ near the earth and arrives near the moon to within a neighborhood of order μ. Similarly to other singular perturbation problems, this uniformly valid solution was obtained by formulating two solutions, one valid near the earth (the "outer solution") and the other valid near the moon (the "inner solution"). The inner solution is expressed in terms of "blown up" variables. The outer and inner solutions are left undetermined by introducing a number of constants; these constants are determined such that the singularities in the outer and inner solutions cancel when they are combined to form the "composite solution."

The basic idea of the method was worked out in its application to the two-fixed center problem with special initial conditions (ref. 3); then the same technique was used in the restricted three body problem with more general initial conditions (ref. 4). One of the most interesting results was the finding that the outer solution <u>must</u> contain a part which is proportional to the small parameter μ, or else it cannot be matched to the inner solution; the outer solution can thus be interpreted as an earth centered Kepler ellipse with a first order correction to take care of the moon's perturbation. In comparing this method with the usual way of "patching conics", it was thus stated that a patched conic method could not be accurate, unless the geocentric ellipse were corrected for the moon's perturbation. The two-variable expansion method was thus offered as an improvement over patched conic methods and it appeared to be (at least initially) equally practical.

This report presents an explanation of the method (in Section 3), based mostly on ref. 4, and the beginning of a somewhat different approach (in Section 4). The claim that this report is an "explanation" is made with all modesty; it is an explanation in the sense that it presents and discusses the major steps of the developments in ref. 4, leaving out many of the laborious details. In this way it is hoped that the reader may gain a full appreciation and understanding of this very interesting method; this report may thus serve as an introduction to the reading of refs. 3 and 4. This explanation is preceded (Section 2) by a general discussion of singular perturbation theory, based mostly on ref. 1 and 2. In particular with respect to this section, and the conjecture and theorem on which the discussion is based, the authors gratefully acknowledge personal communication with Dr. Kevorkian.

In Section 4 the beginning of a slightly different approach to the same problem is presented. Whereas the work by Lagerstrom and Kevorkian is formulated in inertial coordinates, this new approach makes use of rotating coordinates, and the Jacobi Integral in order to solve the problem as a third order system of differential equations.

2. DISCUSSION OF THE TWO-VARIABLE EXPANSION METHOD

The method used by Lagerstrom and Kevorkian to formulate a uniformly valid representation of earth-moon trajectories is that which is used in the singular perturbation problems of boundary layer theory. A singular perturbation problem may be characterized as follows: a differential equation $L(x, u, \varepsilon) = 0$ and boundary conditions $B(u, \varepsilon) = 0$ depend on a small positive parameter ε in such a way that the order or type of L change when $\varepsilon = 0$, while the number of boundary conditions remains unchanged. Thus, if u^0 represents the solution of $L(x, u, 0) = 0$, one may not expect that u approaches u^0 uniformly as $\varepsilon \rightarrow 0$.

Fundamental to the solution of singular perturbation problems is the introduction of certain limits. Consider functions f of ε, positive and continuous in $0 < \varepsilon < A$ and tending to a definite limit as $\varepsilon \rightarrow 0$; introduce a new variable $x_f = \frac{x}{f}$, then a limit on $F(x, \varepsilon)$ is defined as

$$\lim_f F(x, \varepsilon) = \lim_{\varepsilon \to 0} F\{(f x_f), \varepsilon\}, \quad x_f \text{ fixed and } \neq 0.$$

If $f = 1$, the limit is usually called "outer limit," and x the "outer variable", since in the boundary layer problem which motivated this formulation this limit presents a satisfactory approximation in the physical space away from the boundary. An "inner variable" and "inner limit" are obtained in many problems by putting $f = \varepsilon$; the inner limit is an approximation in that region of the physical space where the differential equation changes order (or type) as $\varepsilon \rightarrow 0$. As the inner variable is kept constant, the physical variable x tends to 0 as $\varepsilon \rightarrow 0$; it is as if the problem is discussed in terms of "stretched" or "blown-up" variables. Theoretically of great importance are also the concepts of "intermediate variable" and "intermediate limit," which are intuitively understood as obtained by a function $f(\varepsilon)$, where the order of magnitude $O(f(\varepsilon))$ is in between $O(1)$ and $O(\varepsilon)$. A more rigorous discussion is given by Kaplun in ref. 1.

The formulation of a solution based on inner and outer limit is based on a "matching" of the two limits. But since there is no a-priori reason why the regions of validity of inner and outer limits should overlap, it may seem to be surprising that this has been so successful in many problems. It is here that Lagerstrom and Kaplun have contributed greatly to the understanding of the problem by using the intermediate expansion to bridge the gap. In ref. 5 Erdelyi discusses this in some more detail, but (as here) also in an intuitive manner.

The method by which a uniformly valid solution of singular perturbations is obtained is based on a conjecture and a theorem. The conjecture is: the solution of the limiting differential equation (obtained by subjecting the differential equation to the above defined limiting process) is identical with the limiting approximation of the exact solution. Thus, if an exact solution cannot be obtained directly, one can get an approximation (actually an asymptotic expansion) by solving the limiting differential equation. The validity of this conjecture is supported by a number of problems to which exact solutions are available.

In a singular perturbation problem it will be necessary to combine at least two limiting solutions (i.e. inner and outer) to obtain a uniformly valid solution, that is a solution valid in the entire physical space of the variables. Kaplun's extension theorem bridges the gap which may exist between the regions of validity of the limiting solutions. The formulation of the extension theorem requires the definition of "equivalence classes". Let f and g be functions of ε, positive and continuous and tending to a definite limit as $\varepsilon \longrightarrow 0$, then $f(\varepsilon)$ and $g(\varepsilon)$ belong to the same equivalence class if

$$0 < \lim_{\varepsilon \to 0} \frac{f}{g} < \infty$$

A partial ordering of equivalence classes is defined by

$$\operatorname{ord} f < \operatorname{ord} g \quad \text{if} \quad \lim_{\varepsilon \to 0} \frac{f}{g} = 0.$$

A set S of equivalence classes is convex if, for every ord f and ord g in S, ord $f <$ ord $h <$ ord g implies ord h is in S. Open and closed convex sets of equivalence classes are defined according to the usual definitions of set theory. The extension theorem may now be formulated as:

If an approximation is valid to order ε in a closed set S its domain of validity may be extended to an open convex set S, containing S.

Thus, the inner and outer expansions are valid in larger regions than those for which they were derived. The regions of validity of inner and outer expansions may now overlap or else they may be joined by an intermediate expansion. Whether the inner and outer expansions are matched directly or by the use of an intermediate expansion, the matching is performed by using overlapping regions of validity provided by the extension theorem. It will be seen that in the earth-moon trajectory problem the matching can be performed directly without the use of an intermediate expansion.

The following illustration may be of some help in understanding the meaning of the expansion theorem. In figure 1 the shaded areas in the x, ε space indicate the regions of validity of inner and outer expansions in a problem with singularity at $x = 0$.

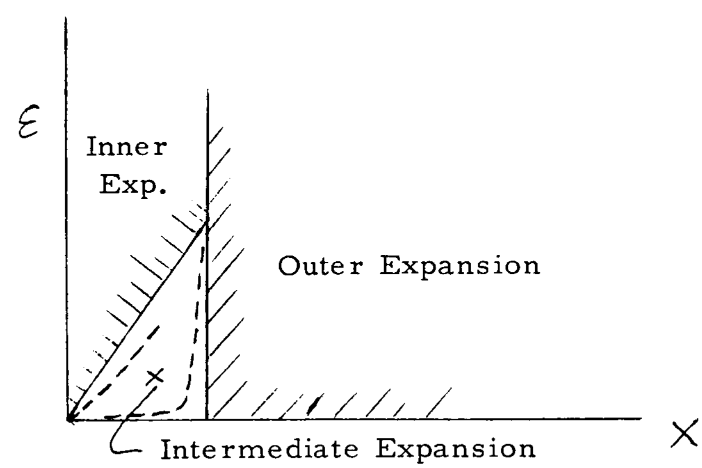


Fig. 1 EXTENSION THEOREM

The outer expansion is valid for a range of x bounded away from zero. The region for the inner expansion shows the typical behavior near the singularity: As ε tends to zero the physical variable x tends to zero also; the inner variable $x_\varepsilon = \dfrac{x}{\varepsilon}$ remains finite. It is clear that for small ε the regions of validity of inner and outer expansions do not overlap. But the expansion theorem provides for small additional regions of validity, indicated by the dashed lines in fig. 1. These regions can now be used to provide overlap with an intermediate expansion (obtained by introducing the intermediate variable $x_f = \frac{x}{f(\varepsilon)}$, ord $\varepsilon < f(\varepsilon) <$ ord 1) and matching can be performed.

The plan for formulating a uniformly valid solution of a singular perturbation problem is now clear. An outer solution of the differential equation is obtained, satisfying some of the boundary conditions. Typically, the boundary conditions near the singularity are neglected, but the outer solution must have as many arbitrary constants as there are neglected boundary conditions. Next, the problem is "blown up" in the region near the singularity by the transformation to inner variables. The boundary conditions which were neglected in the outer solution can now be satisfied by the inner solution, but the other boundary conditions will in general not make sense. Therefore the inner solution is partly indeterminate. To remove this indeterminacy the inner and outer solutions are "matched" as follows. The outer solution is evaluated at the inner region, the inner solution is evaluated at the outer region and these two functions are equated after the introduction of the transformation $x_f = \frac{x}{f(\varepsilon)}$. Finally, a "composite solution" is obtained by adding the inner and outer solutions and subtracting <u>either</u> the inner solution evaluated at the outer region <u>or</u> the outer solution evaluated at the inner region. This either/or condition reflects of course just the matching condition. The matching and the formulation of a composite solution described here is possible when the regions of validity of inner and outer expansions overlap, if this is not the case the same procedures have to be followed on either side of an intermediate expansion.

The extension theorem is the basis for success in matching; the conjecture makes it plausible that the composite solution is uniformly valid, even though the inner and outer solutions themselves are only valid in their respective regions.

The application of these principles to the earth-moon trajectory problem takes the following form. The equations of motion of the planar restricted three body problem (in non-rotating coordinates) are formulated with one of the coordinates, x, as the independent variable. Uniformly valid expressions are sought for the time and the other coordinate as functions of x and the small parameter μ, the earth-moon mass ratio. Near the earth the influence of the moon is seen in the equations of motion as a perturbation (proportional to μ) of the Kepler equations. Clearly, in this problem the singularity is located at $x \to 1$, since near the moon the attraction of the moon itself is the major force. An outer solution is formulated in the physical variables x, t and y; it describes the earth-centered part of the trajectory. An inner solution is formulated in the "blown-up" variables $\bar{x}$, $\bar{t}$ and $\bar{y}$, the differential equations for which show the moon's attraction as the major force. In principle the outer and inner solutions are asymptotic expansions of which the separate terms can be obtained by substituting $t = t_0 + \mu t_1 + \mu^2 t_2 + \ldots$, $y = y_0 + \mu y_1 + \mu^2 y_2 + \ldots$

in the equations of motion, ordering the results according to powers of μ and solving the equations for $t_0, y_0, t_1, y_1, \ldots$ in succession. A major result of Lagerstrom's and Kevorkian's investigation was the finding that, in order to formulate a first order solution, the outer solution must contain the correction of order μ to the earth-centered Kepler trajectory. The reason is that the angular momentum near the moon (for a passage at distance of order μ) is of order μ , and can thus only be defined when terms of order μ are included in the approach trajectory. The matching of inner and outer solutions is performed by equating term by term the results of evaluating the outer solution at $X = 1$ and the inner at $\bar{X} = -\infty$; for this purpose the inner as well as the outer solution are expressed in the inner variable. The results of the matching are the elements of the moon-centered hyperbola and the phase constant of the moon. The composite solution is obtained by adding the inner and outer solutions and subtracting the outer expansion of the inner solution. From the form of inner and outer solutions it is clear that no intermediate solution is required.

In their first paper on the three-body problem (ref. 8) Lagerstrom and Kevorkian treated the problem of two fixed force centers (the Euler problem). They discussed trajectories which leave from the center of the larger mass, the Kepler part of the outbound trajectory being a straight line. The major result was that 1) a uniformly valid solution to order μ could indeed be obtained and 2) the outer solution must contain a correction of order μ in order to be able to determine the constants of the inner solution. Because of the very special initial conditions the outer and inner expansions are of simple form and therefore the principles of the method are clearly demonstrated. In their second paper (ref. 9) they treated the more practical restricted three body problem with arbitrary initial conditions (although restricted to a neighborhood of order μ near the earth). While following the same method in principle, the details of the analysis are somewhat obscured by the added difficulties from the more general initial conditions and the motion of the moon. The following section refers in particular to this paper; it interprets and explains the method by lifting out the essential difficulties and omitting all easily understood details. References 10 and 11 discuss some numerical aspects.

The following section contains an outline and discussion of ref. 9. It is hoped that, by concentrating on the major difficulties, that section, together with the general discussion in this section, will be useful for the better understanding and appreciation of the very interesting method of Lagerstrom and Kevorkian.

3. TWO-VARIABLE EXPANSION METHOD FOR EARTH-MOON TRAJECTORIES

3.1 Equations of Motion; Outer and Inner Variables

In geocentric, non-dimensional, inertial coordinates x, y the equations of motion for the planar restricted three body problem are:

$$\ddot{x} + (1-\mu)\frac{x}{r^3} = \mu f \qquad f = \frac{\xi_m - x}{r^3} - \xi_m \tag{1}$$

$$\ddot{y} + (1-\mu)\frac{y}{r^3} = \mu g \qquad g = \frac{\eta_m - y}{r^3} - \eta_m$$

where

$$r^2 = x^2 + y^2$$

μ is the earth-moon mass ratio, $\mu = \dfrac{M_m}{M_E + M_m}$, and the coordinates of the moon are

$$\xi_m = \cos(t-T), \quad \eta_m = \sin(t-T) \tag{2}$$

T is a phase angle which is to be determined later.

The goal is to formulate uniformly valid expressions (i.e. valid near the earth as well as near the moon) to order μ for trajectories which leave from a neighborhood of order μ near the earth and reach a neighborhood of order μ near the moon.

The outer variables, to be used for the outbound trajectory near the earth, are the physical variables x, y, t. The inner variables will be chosen as

$$\bar{x} = \frac{x - \cos(t-T)}{\mu}, \quad \bar{y} = \frac{y - \sin(t-T)}{\mu} \tag{3}$$

$$\bar{t} = \frac{(t-T) - \tau}{\mu}.$$

This choice assures that the motion near the moon is Keplerian up to and including the first order of μ and that the velocity far from the moon is of the same order (i.e. of order 1) as the velocity far from the earth. The additional phase angle τ is introduced so that $\bar{t}$ can be made to vanish at perilune.

It is interesting to note that if a scale factor of $\mu^{-1/3}$ is used in the definition of $\bar{X}$ and $\bar{Y}$ and the time is left unscaled, the equations of motion in terms of $\bar{X}$, $\bar{Y}$ and $\bar{t}$ after letting $\mu \rightarrow 0$ are the Hill equations; these equations are valid in Hill's region, i.e. a region of order $\mu^{1/3}$ near the moon. If an intermediate solution were required, these equations could provide it. It will be seen that the inner and outer solutions can be matched without using an intermediate solution, although this cannot be expected a priori. Apparently, for the class of trajectories considered here (i.e. coming from a neighborhood of order μ near the moon), the passage through Hill's region is so fast that Hill's equations do not need to be considered.

It will be convenient to introduce the coordinate X as the independent variable; the matching of inner and outer solutions is then done on the basis of distance instead of time. The equations of motion in the outer variables are then

$$
\begin{aligned}
-\frac{t''}{t'^3} + (1-\mu)\frac{X}{r^3} &= \mu f \\
\frac{Y''}{t'^2} - \frac{t''Y'}{t'^3} + (1-\mu)\frac{Y}{r^3} &= \mu g
\end{aligned}
\tag{4}
$$

The equations of motion in inner variables, valid near the moon, are Keplerian up to and including the first order of μ and do not have to be written here. Terms proportional to the first power of μ are not present because of the scaling of the variables and because the moon centered $\bar{x}$, $\bar{y}$ axes are taken parallel to the earth centered x , y axes.

3.2 Outer Expansion

The right hand sides of equs. (4) represent small perturbations due to the moon; near the earth the solution of equ. (4) is thus nearly Keplerian and it will be convenient to specify the initial conditions of the trajectory by giving the values of the Kepler integrals. The integrals to be chosen are the total energy h_e , the angular momentum ℓ_e the location of perigee and the time of perigee passage. In order to reach the neighborhood of the moon, the total energy must be $O(1)$; the initial velocity is thus $O(\mu^{-\frac{1}{2}})$ and, since the trajectory leaves from a neighborhood of order μ near the earth, the angular momentum is $O(\mu^{\frac{1}{2}})$. Without loss of generality the perigee may be taken to be on the x-axis (on the side of the earth opposite to that of the moon). The initial conditions are thus

$$\text{at} \quad x=0 \; : \; h_e = -\rho^2 \tag{5}$$

$$\ell_e = \mu^{1/2}\lambda \tag{6}$$

perigee on x-axis

and the time is specified by requiring that the Keplerian approximation is exact at $x = 0$ to all orders of μ .

Since the angular momentum is of order $\mu^{1/2}$ it is clear that, for the class of trajectories discussed here, y is also of order $\mu^{1/2}$. The asymptotic expansions for t and y may thus be taken to be

$$t(x,\mu) = t_0(x,\mu) + \mu t_1(x) + \ldots \tag{7}$$

and

$$y(x,\mu) = \mu^{1/2} y_{\frac{1}{2}}(x,\mu) + \mu y_1(x) + \ldots \tag{8}$$

The differential equations for t_0, t_1, $y_{1/2}$ and y_1 are found by substituting (7) and (8) into the equations of motion (4) and by ordering the results according to powers of μ. The equations for t_0 and $y_{\frac{1}{2}}$ are of course just the Keplerian equations (equ. 4 with zero in the right handsides) and their solutions do not have to be repeated here. However, one detail must be pointed out. Whenever the parameter μ appears as ($1-\mu$), the nondimensional gravitational constant, it is not subjected to the limit process. Furthermore, the angular momentum constant has been written as $\mu^{1/2}\lambda$ and for these two reasons the parameter μ appears thus in the expressions for the Keplerian part of the trajectory. This seems at first to be in contradiction with the principle of the singular perturbation method according to which the zero-order solution would be independent of the small parameter. Allowing the small parameter to appear in the Keplerian part results in somewhat more convenient expressions. The first part of $t(x,\mu)$ is now written as

$$t_0(x,\mu) = t_{00}(x) + \mu t_{01}(x) \tag{9}$$

If the solution had been started with $t_{00}(x)$, according to a strict application of the limit process it would be necessary to consider a separate "boundary layer" near the earth, because the relative orders of magnitude of the terms in $t_0(x, \mu)$ are different for $x = O(1)$ and $x = O(\mu)$. This nonuniformity has nothing to do with the moon's perturbation and is taken care of by letting μ appear in the Keplerian solution.

The equations for the first order corrections t_1 and y_1 are:

$$- \frac{t_1''}{t_{00}'^3} + \frac{3 t_{00}'' t_1'}{t_{00}'^4} = f_0 \tag{10}$$

$$\frac{y_1''}{t_{00}'^2} - \frac{t_{00}'' y_1'}{t_{00}'^3} + \frac{y_1}{x^3} = g_0 \tag{11}$$

with $f_0 = \{f(x)\}_{\mu=0}$ and $g_0 = \{g(x)\}_{\mu=0}$.

Because the initial conditions have been chosen such that the Kepler solution is exactly valid at $x = 0$, the initial conditions for t_1 and y_1 are simply

$$t_1(0) = y_1(0) = 0 \quad \text{and} \quad t_1'(0) \quad \text{and} \quad y_1'(0).$$

3.3 First Order Corrections in Outer Expansion

The first order integrals of (10) and (11) are easily obtained as

$$-t_1' = t_{00}'^3 \int_0^x f_0(\xi)\, d\xi \tag{12}$$

and

$$x y_1' - y_1 = t_{00}' \int_0^x \xi\, t_{00}'(\xi)\, g_0(\xi)\, d\xi \tag{13}$$

In principle t_1 and y_1 are thus obtained by quadratures but so far no analytic expressions for t_1 and y_1 have been found. The functions f_0 and g_0 are unbounded for $X \longrightarrow 1$ and their behavior near $X = 1$ can be studied by expressing the several parts of f_0 and g_0 in Taylor series near $X = 1$. The results are

$$f_0(x) = \frac{U}{(1+U^2)^{3/2}}\left[\frac{U^2}{(1-x)^2} + \frac{1}{(1-x)}\right] + \Phi(x) \tag{14}$$

$$g_0(x) = \frac{-1}{(1+U^2)^{3/2}}\left[\frac{U^2}{(1-x)^2} + \frac{1}{(1-x)}\right] + \Gamma(x) \tag{15}$$

where $\Phi(x)$ and $\Gamma(x)$ are the regular parts of f_0 and g_0, and

$$U = \sqrt{2(1-\rho^2)} = \frac{1}{t_{00}'(1)} \tag{16}$$

which is the x velocity of the Keplerian trajectory at $x = 1$.

Using (14) and (15) the first order corrections to the Keplerian part of the outer expansion may be written as

$$t_1(x) = -\frac{U}{(1+U^2)^{3/2}}\int_0^x t_{00}'^{3}(\xi)\left[\frac{U^2}{1-\xi} - \log(1-\xi) + \frac{(1+U^2)^{3/2}}{U}\int_0^\xi \Phi(\xi)\,d\xi\right]d\xi \tag{17}$$

$$y_1(x) = -\frac{x}{(1+U^2)^{3/2}}\int_0^x \frac{t_{00}'(\xi)}{\xi^2}\left\{\int_0^\xi \xi t_{00}'(\xi)\left[\frac{U^2}{(1-\xi)^2} + \frac{1}{1-\xi} - (1+U^2)^{3/2}\Gamma(\xi)\right]d\xi\right\}d\xi \tag{18}$$

Since (at least to this time) no analytic solutions for t_1 and y_1 have been found, the complete trajectory can only be computed by evaluating the quadratures numerically. Clearly, this causes numerical difficulties because of the singular behavior near $x = 1$. It is of some help in establishing a computer program based on this method that $t_1(x)$ and $y_1(x)$ depend on only one parameter, namely the total energy $-\rho^2$. The corrections could thus be computed and tabularized once and for all. Also, near $x=1$ t_1 and y_1 may be expressed much more simply as

$$t_1 = (1+u^2)^{-3/2} \log(1-x) + \gamma(\rho) + O(1) \tag{19}$$

$$y_1 = (1+u^2)^{-3/2} \log(1-x) + \delta(\rho) + O(1) \tag{20}$$

where γ and δ are functions of the total energy alone. Unfortunately, γ and δ become unbounded as $\rho \longrightarrow 1$, that is for the minimum energy trajectories. This difficulty has been treated in detail in ref. 6.

Equ. (19) and (20), and particularly the functions γ and δ, play an important role in the matching of outer and inner expansions.

3.4 The Inner Expansion

The equations in the inner variables $\bar{x}$, $\bar{y}$ and $\bar{t}$ have μ only to the second and higher powers. Since the present purpose is to develop a solution to first order in μ the moon centered part of the trajectory is thus Keplerian and, in particular, hyperbolic. It will be convenient to characterize this hyperbola by the four constants

u_1, the $\bar{x}$ component of velocity at $\bar{x} = -\infty$

V_1, the $\bar{y}$ component of velocity at $\bar{x} = -\infty$

$K_1 = \dfrac{AV_1 - Bu_1}{u_1}$ related the direction of the asymptote

and $\bar{t} = 0$ at perilune.

In the definition of K_1 ,

$$A = \bar{a}\,\bar{e}\cos\bar{\theta} \qquad B = \bar{a}\,\bar{e}\sin\bar{\theta}$$

$\bar{a}$ is the semimajor axis, $\bar{e}$ the eccentricity and $\bar{\theta}$ is the counter-clockwise angle between the x axis and the apse line of the hyperbola. The expressions $\bar{y}(\bar{x})$ and $\bar{t}(\bar{x})$ do not have to be given here (since they are Keplerian) except as they are needed for the matching of inner and outer solutions. For this purpose their values as $x \longrightarrow -\infty$ are needed. These are

$$\bar{y} = \frac{V_1}{U_1}\,\bar{x} - \frac{AV_1 - BU_1}{U_1} \tag{21}$$

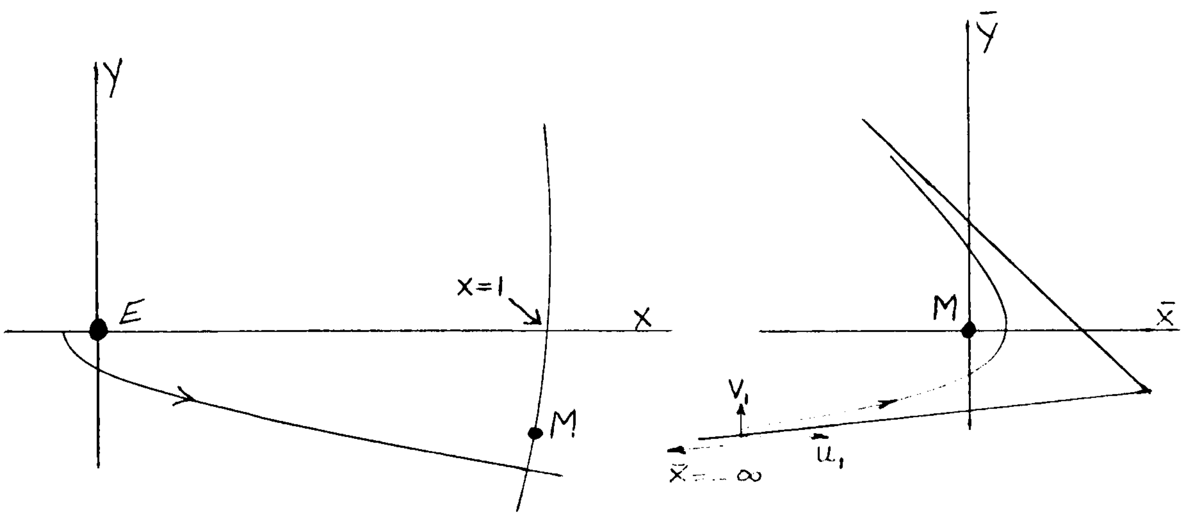
$$\bar{t} = \frac{\bar{x} - A}{U_1} + \bar{a}^{3/2} \log \frac{-2\bar{x}}{U_1 a^{3/2}\bar{e}} \tag{22}$$

as follows readily from the equations of hyperbolic motion (most conveniently by letting the eccentric anomaly approach $-\infty$).

3.5 Matching of Inner and Outer Solutions

The purpose of matching the inner and outer expansions is to determine the constants of the moon,centered hyperbola, thereby also relating the singularities in the two expansions in such a way that they cancel each other in the composite solution. Because the singularities are logarithmic in nature in the inner as well as the outer solutions, such matching can apparently be achieved without the use of an intermediate expansion.

The geometry of the matching is illustrated in Fig. 2, as much as it can be illustrated. The part of the figure related to the inner expansion is drawn in the scaled coordinates $(\bar{x}, \bar{y})$ and must be thought as infinitely small in comparison with the figure for the outer expansion. It may be remarked that this matching is strictly analytical, whereas the "patching" of conics is strictly geometrical. A direct comparison of the two methods is therefore difficult; such comparison should be based on the final numerical results.



OUTER EXPANSION

$$t = t_{00}(X) + \mu t_{01}(X) + \mu t_1(X)$$

$$Y = \mu^{1/2} Y_{\frac{1}{2}}(X) + Y_1(X)$$

INNER EXPANSION

$$t = T + \mu \bar{t} + \tau$$

$$y = \mu \bar{y} + \sin(t - T)$$

Figure 2 GEOMETRY OF MATCHING

The matching is performed by evaluating the outer expansion at $X = 1$ and equating the result term by term to the inner expansion evaluated at $\bar{X} = -\infty$, both expansions being expressed in the inner variable. (The important thing is that both expansions are expressed in the same variable;

the present choice of inner variable is simply for the sake of convenience.)

The part of the outer expansion identified with t_{oo} is evaluated near $X = 1$ by writing two terms of its Taylor expansion at $X = 1$:

$$t_{oo}(x) = t_{oo}(1) + (x-1)\,t'_{oo}(1) = t_{oo}(1) + (x-1)\,\mathcal{U}^{-1}$$

using equ. (16). The inner variable is introduced by $X = \mu\bar{X} + \cos(t-T)$ from equ. (3), and if it is assumed that $(t - T)$ is small (as it will be shown to be), there results

$$t_{oo}(x) = t_{oo}(1) + \mu\left\{\bar{x}\,\mathcal{U}^{-1} - \frac{1}{2}(t-T)^2\,\mathcal{U}^{-1}\right\} \qquad (23)$$

The introduction of the inner variable into the expression for t_1 (equ. 19) is taken care of by putting $(1-X) = \mu\bar{X}$, the term $\cos(t-T)$ in equ. (3) being put equal to unity with enough accuracy since t_1 is multiplied by μ. By combining equs. (7), (9), (19) and (23), the outer expansion evaluated near $X = 1$ and expressed in the inner variable is thus

$$\begin{aligned} t(\bar{x}) = {} & t_{oo}(1) + \mu\bar{x}\,\mathcal{U}^{-1} - \frac{1}{2}(t-T)^2\,\mathcal{U}^{-1} \\ & + \mu\left\{t_{o1}(1) + (1+\mathcal{U}^2)^{-3/2}\left[\log(-\bar{x}) + \log\mu\right] + \gamma(\rho)\right\} \end{aligned} \qquad (24)$$

From equ. (3) and (22) follows for the inner expansion evaluated at $X = -\infty$

$$\begin{aligned} t(\bar{x}) = T + \tau + \mu\Big\{ & \bar{x}\,\mathcal{U}_1^{-1} - A\,\mathcal{U}_1^{-1} + \bar{a}^{-3/2}\log(-\bar{x}) \\ & + \bar{a}^{-3/2}\log\frac{2}{\gamma\,\mathcal{U}_1\,\bar{a}^{3/2}\,\bar{e}}\Big\} \end{aligned} \qquad (25)$$

Now, if the phase angle T is chosen to be composed of several parts according to powers of μ as follows,

$$T = T_0 + \mu^{1/2} T_{1/2} + \mu T_1 \qquad (26)$$

the third term in equ. (24) is to first order in μ, $-\frac{1}{2}\mu T_{\frac{1}{2}}^{2} \mathcal{U}^{-1}$ and the two expressions for $t(\bar{X})$ can be made identical by making the following choices for T_0, $\mathcal{U}_1$, and τ.

$$T_0 = t_{00}(1) \qquad (27)$$

$$\mathcal{U}_1 = \mathcal{U}$$

$$\tau = -\mu^{1/2} T_{\frac{1}{2}} - \mu \Big\{ T_1 + \tfrac{1}{2} T_{\frac{1}{2}}^{2} \mathcal{U}^{-1} - t_{01}(1) - \qquad (28)$$

$$-(1+\mathcal{U}^2)^{-3/2} \log \mu - \gamma(\rho) - A\mathcal{U}_1^{-1} + \bar{a}^{3/2} \log \frac{2}{\mathcal{U}_1 \bar{a}^{-3/2} \bar{e}} \Big\} \qquad (29)$$

Note that this implies that $(1+\mathcal{U}_1^{2})^{-1} = \bar{a}$; this will be confirmed by the matching of the expansions for Y.

From equ. (8), (20) and (3) follow for the outer expansion of Y evaluated near $X = 1$ and expressed in the inner variable $\bar{X}$,

$$Y(\bar{X}) = \mu^{1/2} Y_{1/2}(1,0) + \mu\left[(1+\mathcal{U}^2)^{-3/2}\left(\log(-\bar{X}) + \log \mu\right) + \delta(\rho)\right] \qquad (30)$$

Since the Keplerian part of this expression is multiplied with $\mu^{1/2}$, its value near $X = 1$ is obtained simply by substituting $X = 1$; no Taylor expansion need be used here because the second term would be proportional to $\mu^{3/2}$.

From equ. (3) and the expression for the inner expansion, equ. (21) follows for the inner expansion evaluated near $\bar{X} = -\infty$,

$$y(\bar{x}) = \mu\left[\frac{V_1}{U_1}\bar{x} - K_1\right] + (t-T)$$

where it is again assumed that $(t-T)$ is small, so that $\sin(t-T) = (t-T)$. This assumption is shown to be valid (at least to order μ) by equ. (25) and (29). If then the expression for $t(\bar{x})$ near $x=1$ (equ. (25)), and the evaluation of τ (equ. (29)), which followed from the matching of the expressions for the time, are used, there follows for $y(\bar{x})$,

$$y(\bar{x}) = -\mu^{1/2}T_{\frac{1}{2}} + \mu\Big[\frac{V_1}{U_1}\bar{x} - K_1 + \bar{x}U^{-1} + \bar{a}^{3/2}\log(-\bar{x}) + (1+U^2)^{-3/2}\log\mu + t_{01}(1) - \tfrac{1}{2}T_{\frac{1}{2}}^{2}U^{-1} + \gamma(\rho) - T_1\Big] \qquad (31)$$

The expressions (30) and (31) are made identical by the following choice of the constants $T_{\frac{1}{2}}$, V_1, K_1 and T_1.

$$T_{\frac{1}{2}} = -Y_{\frac{1}{2}}(1,0) \qquad (32)$$

$$V_1 = -1 \qquad (33)$$

$$-K_1 = \tfrac{1}{2}T_{\frac{1}{2}}^{2}U^{-1} - t_{01}(1) + T_1 + \delta(\rho) - \gamma(\rho) \qquad (34)$$

T_1 is arbitrary

The result $V_1 = -1$ confirms the expression $\bar{a} = (1+U_1^2)^{-1}$ which was necessary for the time-matching since for the moon centered hyperbola $\bar{a} = \frac{1}{2\bar{h}} = (V_1^2 + U_1^2)^{-1}$. With equs. (28), (33) and (34) the moon centered hyperbola is now determined, the constants $\bar{h}$, $\bar{l}$, $\bar{p}$ and $\bar{q}$ being expressed as

$$
\begin{aligned}
2\bar{h} &= 1 + u_1^2 = 3 - 2\rho^2 \\
\bar{\ell} &= K_1 u_1 \\
\bar{p} &= K_1 u_1^2 + (1+u^2)^{-1/2} \\
\bar{q} &= K_1 u_1 - u_1 (1+u_1^2)^{-1/2}
\end{aligned}
\tag{35}
$$

These four constants are really equivalent to three integrals because

$$\bar{p}^2 + \bar{q}^2 = 2\bar{\ell}^2\bar{h} + 1$$

so that a fourth integral is still needed. This is provided by the condition that $\bar{t} = 0$ at perilune. This condition is satisfied by the proper choice of the phase angle T and the origin of the inner variable $\bar{t}$ which are determined by equ. (27) for T_c, equ. (32) for $T_{1/2}$, and equ. (29) for τ. The constant A which is needed in equ. (29) is simply

$$A = -\frac{\bar{q}}{2\bar{h}}$$

It is a fortunate circumstance that T_1, the part of the phase angle T which is proportional to μ, is arbitrary. T_1 influences K_1, and thereby the angular momentum $\bar{\ell} = K_1 u_1$. With the hyperbola's total energy determined by u_1, the perilune distance can thus be adjusted by changing the angular momentum through T_1.

It may now be noted that the phase angle T (apart from the arbitrary contribution μT_1) and two of the hyperbolic constants depend only on the Keplerian part of the outbound trajectory. As a matter of fact, Lagerstrom and Kevorkian derived T_c and $T_{1/2}$ in the very beginning of their analysis and on the basis of the outbound Kepler trajectory alone. For the purpose of this presentation of their analysis it was felt that the modification in which T_c and $T_{\frac{1}{2}}$ are derived from the matching conditions is a little more in line with the general principle of the method of singular perturbations; this principle being the determination of certain constants, which leave the inner and outer expansions indeterminate, from matching conditions.

Furthermore, it is noted that the first order corrections of the outbound trajectory enter into the matching conditions only through the functions $\gamma(\rho)$ (in the determination of τ) and $(\delta(\rho) - \gamma(\rho))$ (in the determination of K_1). The functions γ and δ become unbounded as $\rho \longrightarrow 1$, i.e. for minimum energy trajectories, but the difference $(\delta - \gamma)$ was shown to be finite (ref. 11). The difference $[\delta - \gamma]$ may be interpreted as the correction of K_1 , required if K_1 were determined on the basis of the Keplerian trajectory alone. Since K_1 is the $\bar{y}$ -intercept of the approach asymptote of the moon-centered hyperbola at $X = 1$, it has been claimed that $(\delta - \gamma)$ is a measure of the error made in the usual methods of "patched-conic" computations; $(\delta - \gamma)$ is then simply the miss-distance of the approach trajectory. Because of the basis difference in the two methods (which has been pointed out earlier in this report: patching conics is geometric, while matching inner and outer expansions is analytic) a comparison on the basis of $(\delta - \gamma)$ tends to come out unfair for the patched-conic method. It would be interesting to see how the corrections t_1 and y_1 contribute to the outbound trajectory near its intersection with the moon's sphere of influence.

And if a thorough comparison of the two methods were desired, it should of course be based on final numerical results for representative trajectories computed by both methods. Lagerstrom and Kevorkian themselves have not provided such a comparison, except by pointing out that $(\mathcal{Y} - y)$ is a measure of the patched conic error; in ref. 5 there are comparisons with exact (i.e. numerically integrated) trajectories, but whether or not the results say much for the two-variable expansion method depends mostly on what kind of errors one is willing to except.

3.6 The Composite Solution

The outer and inner solutions have been formulated and their singular behavior has been identified. By matching these two solutions in their overlapping region of validity the phase angle and the constants of the moon centered hyperbola have been determined. To complete the work a composite solution must be formulated. According to the singular perturbation theory the composite solution is obtained by adding the outer and inner solutions and subtracting their common part. That common part is just the inner solution evaluated in the outer region (that is for at $\bar{x} \longrightarrow -\infty$), or the outer solution evaluated in the inner region (that is for $x \longrightarrow 1$); these two evaluations are identical because that was just the condition for matching. Here it is convenient to use the inner solution evaluated for $\bar{x} \longrightarrow -\infty$.

According to equ. (8) the outer solution is

$$y(x, \mu) = \mu^{1/2} y_{1/2}(x, \mu) + \mu y_1(x)$$

where $y_{1/2}$ and y_1 are known functions, y exhibiting a singularity for $x \to 1$.

According to equ. (3) the inner solution is

$$y(x,\mu) = \eta_m + \mu \bar{y}(\bar{x})$$

where the moon coordinate $\eta_m = \sin(t-T)$, $\bar{x} = \dfrac{x-\xi_m}{\mu}$ with $\xi_m = \cos(t-T)$ and $\bar{y}(\bar{x})$ is the equation of the moon-centered hyperbola. According to equ. (3) and (21), the inner solution evaluated for $\bar{x} \rightarrow -\infty$ is

$$y(x,\mu)_{\bar{x}\rightarrow-\infty} = \eta_m + \alpha(\bar{x})$$

with $\bar{\alpha}(\bar{x}) = \dfrac{V_1}{u_1}\bar{x} - \dfrac{K_1}{u_1}$

The composite solution for y is thus

$$y(x,\mu) = \mu^{1/2} y_{1/2}(x,\mu) + \mu y_1(x) + \mu\left[\bar{y}(\bar{x}) - \alpha(\bar{x})\right]$$

and in the same way the composite solution for $t(x,\mu)$ is found to be

$$t(x,\mu) = t_0(x,\mu) + \mu t_1(x) + \mu\left[\bar{t}(\bar{x}) - \beta(\bar{x})\right] \qquad (36)$$

where $$\beta(\bar{x}) = \frac{\bar{x}-A}{u_1} + \bar{a}^{3/2} \log \frac{-2\bar{x}}{u_1 \bar{a}^{3/2}\bar{e}} \qquad (37)$$

If analytical expressions for $t_1(x)$ and $y_1(x)$ were available it would be observed that their singularities are cancelled by the singularities of the expressions in square brackets; this is for instance the case in the analysis for the two-fixed center problem (ref. 3). In the absence of analytic expressions for t_1 and y_1, the singularities must cancel numerically. Now, to determine just the geometry of the moon centered hyperbola (determined by the constants in equ. (35), the functions t_1 and y_1 themselves are not needed, only the function $(\delta - \gamma)$ is. $(\delta - \gamma)$ depends on the initial condition $-\rho^2$ only and can be computed and tabulated once and for all. However, if the time-dependency and the entire trajectory is needed, the functions t_1 and y_1, as well as the expressions in the square brackets of equs. (36) and (37) must be computed and their singularities made to cancel numerically; this may be expected to cause some numerical difficulties.

4. THE OUTER EXPANSION IN ROTATING COORDINATES

In the previous section it was shown that the application of singular perturbation theory results in a uniformly valid solution to first order of for a certain class of trajectories in the restricted three body problem. In principle this is a satisfactory solution, but practically there are some difficulties because this solution is left in terms of quadratures which must be numerically integrated. Furthermore, since the formulation was carried out in a non-rotating coordinate system one may ask whether a formulation in a different coordinate system would be more advantageous.

Therefore, in conclusion, the following items are cited as possibly leading to improvements or analytical simplifications for this type of first order solution:

1) to obtain analytical approximations for the quadratures which depend on some parameter of the zero-order ellipse (in this case the energy);

2) to represent the problem in a rotating coordinate system as a third order system of differential equations by making use of the Jacobi Integral.

An investigation of the second recommendation has been initiated and in what follows the results for the outer solution are outlined in terms of quadratures. As a result of this investigation it was found that in addition to the choice of a rotating frame of reference the choice of polar coordinates was a decided advantage for the following reasons:

1) The solution for time is obtained from the first order differential equation provided by the Jacobi Integral;

2) The occurrence of elliptical integrals in the zero-order solution for the time is avoided when the radius is used as independent variable;

3) A solution in polar coordinates readily lends itself to extension to three dimensions.

The details of this analysis follow.

In the planar restricted three body problem assume a non-rotating earth-centered coordinate system with axes X, Y parallel to some inertial axes and let the earth-moon distance equal 1 while the masses of the earth and moon are $1-\mu$ and μ respectively and the gravitational constant $k^2=1$. The Lagrangian for a massless particle at (x, y) is from Reference 7:

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1-\mu}{r} + \frac{\mu}{r_2} - \mu(x \cos t + y \sin t) \tag{38}$$

In this system the moon rotates with angular velocity $\omega = 1$ and the transformation to a rotating coordinate system X^*, Y^* with the moon at unit distance on the X^* axis is:

$$\begin{aligned} x^* &= x \cos t + y \sin t \\ y^* &= -x \sin t + y \cos t \end{aligned} \tag{39}$$

where in polar coordinates relative to the rotating coordinate system:

$$\begin{aligned} x^* &= r \cos \theta^* \\ y^* &= r \sin \theta^* \end{aligned} \tag{40}$$

and $r = r^*$. The Lagrangian in relative polar coordinates becomes:

$$L^* = \frac{1}{2}\left[\dot{r}^2 + (r\,\dot{\theta}^*)^2\right] + r^2\,\dot{\theta}^* + \frac{1-\mu}{r} + \frac{\mu}{r_2} - \mu\, r \cos \theta^* \tag{41}$$

where: $r_2 = \left[1 + r^2 - 2r \cos \theta^*\right]^{1/2} = \left[(r-1)^2 - \frac{(y^*)^2}{r}\right]^{1/2}$

Since L^* is time independent there exists an integral of the equations of motion known as the Jacobi integral which is equal to the Jacobi Constant C' for the relative energy. Thus the expression for the relative velocity becomes:

$$\dot{r}^2 + r^2 (\dot{\theta}^*)^2 = r^2 + \frac{2(1-\mu)}{r} + \frac{2\mu}{r_2} - 2\mu \, r \cos \theta^* - C' \tag{42}$$

An asymptotic series solution of the following form is to be obtained:

$$\begin{aligned} t &= t_o(r) + \mu \, t_1(r) + 0(\mu^2) \\ \theta^* &= \theta_o^*(r) + \mu \, \theta_1^*(r) + 0(\mu^2) \end{aligned} \tag{43}$$

where $\mu \backsim .01$. The zero order solution is a two body ellipse relative to the earth with elements a and b, e, i, ω', Ω, Υ and constant angular momentum ℓ_o and energy h_o. It will be assumed that the initial conditions are taken at the perigee. Then the solution for t is essentially a first order approximation to a "Kepler's Equation" for a special class of lunar trajectories in the planar restricted three body problem and t_o is exactly Kepler's equation for the two body problem:

$$t_o = \sqrt{\frac{a^3}{1-\mu}} \left(\cos^{-1} \left(\frac{a - r}{ae}\right) - e \sqrt{1 - \left(\frac{a - r}{ae}\right)^2} \right) \tag{44}$$

and θ_o^* is given by:

$$\theta_o^* = \cos^{-1} \left[\frac{a(1-e^2) - r}{re} \right] - t_o - \Psi_i$$

where Ψ_i is an initial phase angle between the semi major axis and the X^* axis. Such a zero order solution is valid since Lagerstrom and Kevorkian, Ref. 4, have shown that within a small neighborhood of the earth of $0(\mu^\alpha)$ the motion is Keplerian up to order $\mu^{1 + 3\alpha}$. Hereafter the subscript zero refers to values for the zero order solution and the subscript i refers to the initial conditions.

The Lagrange equation $\left[L\right]_\theta = 0$ provides the following expression for the change in the total angular momentum:

$$\frac{d}{dt}\left(r^2\dot{\theta}^* + r^2\right) = \mu\left[\frac{-\ r \sin\theta^*}{(r_2)^3} + r \sin\theta^*\right] \tag{45}$$

Integrating for a first order approximation gives:

$$r^2\,\dot{\theta}^* + r^2 - \ell_o = \mu\int_{r_i}^{r}\left(\frac{-r\sin\theta o^*}{\left(r_{2(o)}\right)^3} + r\sin\theta_o^*\right)\frac{dt_o}{dr}\,dr \tag{46}$$

Clearly the integrand in equ. (46) is expressible as a function of r through eqs. (44). However due to the transcendental nature of the resulting expression for the integrand an analytical integration cannot be obtained directly. Instead an approximation for the integral dependent on certain parameters of the zero order solution can be determined and exercising choice as to the form of the approximation will allow some simplification of the solution for t. Now $\dot{\theta}^*$ becomes:

$$\dot{\theta}_o^* + \mu\dot{\theta}_1^* = \frac{\ell_o}{r^2} - 1 + \mu\,P(r) \tag{47}$$

where the approximation for the integral has been incorporated in $P(r)$.

Now the Jacobi Integral, equ. (42) provides a first order differential equation for t after substituting for $\dot{\theta}^*$:

$$(t_o')^2 + 2\,\mu\, t_o'\, t_1' = \frac{r^2}{(2\,\ell_o - {}^2C)\, r^2 + 2(1-\mu)\, r - \ell_o^2 + 2\,\mu\, r^2 \left(\frac{1}{r_{2(0)}} + \right.}$$

$$\left. P(r)\,(r^2 - \ell_o) - \Delta - r \cos \theta_o^* \right) \tag{48}$$

and

$$t_o' = \frac{r}{\sqrt{(2\,\ell_o - {}^2C).r^2 + 2(1-\mu)\, r - \ell_o^2}}$$

where ${}^2C = 2\,(\ell_o - h_o)$ is the energy constant for a two body orbit relative to a rotating reference frame and $\Delta = C' - {}^2C$. Note that in equ. (48) both $\frac{1}{r_2}$ and $P(r)$ become unbounded as $r_2 \longrightarrow 0$; however, the combination of these terms should remain bounded insuring that $\frac{dr}{dt}$ is bounded near the moon.

Similarly $\theta^{*\,'}$ is obtained from equ. (48):

$$\theta_o^{*\,'} + \mu\theta_1^{*\,'} = \frac{\ell_o}{r^2}\; t_o' - t' + \mu\, \frac{\ell_o}{r^2}\; t_1' + \mu\, P(r)\, t_o' \tag{49}$$

where again the prime denotes differentiation with respect to r.

This completes the outline of the outer solution. A similar investigation of the inner solution and the results of matching the solutions will be final deciding factors in the determination of the practicality of this approach.

5. CONCLUSION

Interpreting the restricted three body problem as a singular perturbation problem results in a uniformly valid solution to first order in the small parameter μ for earth-moon trajectories. This solution can be thought of as being composed of an "outer solution," valid near the earth and an "inner solution," valid near the moon. The outer and inner solutions are matched in their common region of validity by determining certain constants (i.e. the initial phase angle of the moon and the elements of the moon-centered hyperbola) in such a way that the singularities which appear in the inner and outer solution vanish in the construction of the composite solution. The matching constants are expressed in terms of the initial conditions, with the exception of a part of order μ in the phase angle which can be chosen arbitrarily and can thus be used to adjust the lunar perigee distance.

It has been shown that the outer solution must necessarily contain a part that is proportional to the small parameter μ in order to make the match with the inner solution possible. A posteriori this conclusion could have been anticipated from a consideration of the order of magnitude of the angular momenta of inner and outer solutions. The need for this first order correction to the earth-centered outbound ellipse seems to explain why the usual patched conic methods (in which such a correction is not made) must be inaccurate. But such a statement must be made with some care, since in the two methods the matching is performed on a very different basis. In the two-variable expansion method the outer solution is evaluated at the moon's distance and equated to the inner solution evaluated far away from the moon, but far away in terms of the "blown-up" inner variable. Although this procedure makes good sense analytically, it is hard to see what it means geometrically. On the other hand, in the patched-conic method the earth centered ellipse (an uncorrected outer solution) is evaluated at the sphere of influence of the moon and equated to the moon centered hyperbola (the inner solution, but in physical variables) at that point. To make a sound comparison of the two methods, it should be based on the final numerical results, or at least one should determine how much the first order correction of the outer solution contributes to the Kepler ellipse up to the moon's sphere of influence.

The composite solutions, in particular the first order correction, is left in the form of quadratures for which no analytic expressions has been found yet. Therefore, although in theory the singularities of outer and inner so solutions cancel, the singularities must be evaluated numerically. This will cause numerical problems if the entire trajectory is to be known as a function of the time. On the other hand, if it is sufficient to just know the elements of the moon centered hyperbola, the quadratures need not be evaluated entirely. Only the parts of the first order correction indicated by $\gamma(\rho)$ and $\delta(\rho)$ are required, and in particular their difference $(\delta - \gamma)$. These functions depend only on the total energy $-\rho^2$ and can be evaluated once and for all for any interesting range of energies. There is an additional difficulty since δ and γ tend to infinity for minimum energy trajectories, but even there the difference $(\delta - \gamma)$ remains finite.

These difficulties may limit somewhat the practicality of the methods depending on how much trouble one would want to go through to write a computer program that evaluates the quadratures. Even so the method is of great interest and a similar development may be attempted along some different approach. Such a different approach is given in Section 4.

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SUMMARY

This report presents a discussion of Lagerstrom and Kevorkian's two-variable expansion method for the computation of lunar trajectories. It discusses the general background of the method in terms of singular perturbation theory and the major steps in the development of a uniformly valid solution for earth-moon trajectories. Also a slightly different approach to the same problem is outlined.

KEY WORDS

Lunar Trajectories

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